Exceptional collections on toric Fano threefolds and birational geometry

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Abstract

Bernardi and Tirabassi show the existence of a full strong exceptional collection consisting of line bundles on smooth toric Fano 3-folds under assuming Bondal's conjecture, which states that the Frobenius push-forward of the structure sheaf $\mathcal{O}_X$ generates the derived category $D^b(X)$ for smooth projective toric varieties X.

In this article, we show Bondal's conjecture for smooth toric Fano 3-folds and also improve their result, using birational geometry.

1 Introduction

A full strong exceptional collection is a building block of the derived category $D^b(X)$ of coherent sheaves on a smooth projective variety X. Such a collection rarely exists, but if it exists, the derived category $D^b(X)$ is equivalent to the derived category $D^b(\mathrm{mod}\, A)$ of the module category $\mathrm{mod}\, A$ of a finite dimensional algebra A.

For any smooth toric DM stack X, Kawamata shows that there is a full, but not necessarily strong, exceptional collection consisting of coherent sheaves on X [Ka06]. Furthermore full strong exceptional collections on toric varieties are studied by many people (cf. [BT10, CM10, CM, CRM09, Cr09, DLM10, HP08, LM10, IU09]).

We can define an endomorphism F_m ($m \in \mathbb{Z}_{>0}$) called *Frobenius map* on any toric varieties over a field of any characteristic (some people also call it a *multiplication map*). It is also known that for smooth complete toric varieties X, $F_{m*}\mathcal{O}_X$ splits into line bundles and Thomsen [Th00] finds an algorithm to compute the set of all direct summands of it. We denote the set by $\mathfrak{D}_X$ for a sufficiently divisible integer m. On the other hand, Bondal's conjecture predicts that the set $\mathfrak{D}_X$ classically generates the derived category $D^b(X)$. So sometimes, for instance in the case $|\mathfrak{D}_X| = \mathrm{rank}\, K(X)$, it becomes a candidate of a full strong exceptional collection consisting of line bundles on smooth projective toric varieties X.

Bernardi and Tirabassi show the existence of such collections on all eighteen smooth toric Fano 3-folds by using Frobenius maps [BT10]. Using

birational geometry, we obtain a stronger result. Precisely we show the following.

Theorem 1.1. *Precisely for sixteen smooth toric Fano* 3*-folds* X *over* $\mathbb{C}$*, the set* $\mathfrak{D}_X$ *becomes a full strong exceptional collection (in an appropriate order). For the rest two cases,* (4) *and* (11) *in Theorem 3.1, there is a unique proper subset of* $\mathfrak{D}_X$ *which becomes a full strong exceptional collection.*

Note that this theorem implies that Bondal's conjecture is true for smooth toric Fano 3-folds. The strategy to prove Theorem 1.1 is as follows;

Step 1. Let $f: X \to Y$ be an extremal birational contractions between smooth toric Fano 3-folds. Assume that $\mathfrak{D}_X$ forms a full strong exceptional collection. Then so is $\mathfrak{D}_Y$. This is done by Lemmas 5.1 and 6.4.

Step 2. By Step 1, it is enough to show that $\mathfrak{D}_X$ forms a full strong exceptional collection only for (birationally) maximal Fano 3-folds X, namely in (11), (17) and (18) in Theorem 3.1. Unfortunately, in the case (11), $\mathfrak{D}_X$ does not form a strong exceptional collection. Instead we can find a subset $\mathfrak{D}_{\mathrm{nef}}$ of $\mathfrak{D}_X$ which becomes a full strong exceptional collection. Then, as in Step 1, an inductive argument works in the case X in (11).

Step 3. To check the strongness of the chosen set in Step 2, it is enough to check the dual of line bundles in the set are nef (Lemma 3.8). This is easily done by observing Figure 5. To check the fullness in Step 2, we prove Bondal's conjecture in our situation by rather tedious, but elementary calculation. This step is done in §4 and §5.

In [BT10], Bernardi and Tirabassi check a similar statement to Theorem 1.1 only in the cases (9), (11), (14), (15) and (16) separately, since the existence of a full strong exceptional collection, not necessarily coming from $\mathfrak{D}_X$, was already known in the rest cases. In their proof of the strongness, they use rather tedious argument, firstly given by [BH09]. Moreover they show the fullness by assuming Bondal's conjecture.

Dubrobin conjectures that for a smooth projective variety X, the quantum cohomology of X is semi-simple if and only if X is a smooth Fano variety with a full exceptional collection. Although his conjecture turns out to be wrong, it is still believed that there is a relationship between the existence of full exceptional collections on X and its quantum cohomology (cf. [Ba04].) Furthermore several people conjecture that every smooth toric Fano variety has a full strong exceptional collection consisting of line bundles [BH09, CM10]. These are the motivations of our study.

It should be pointed out that there is a smooth projective toric (not Fano) surface which does not possess any full strong exceptional collections

consisting of line bundles ([HP06, HP08]), and it is also worthwhile to mention that there is a smooth toric Fano variety X such that we cannot choose full strong exceptional collections from the set $\mathfrak{D}_X$ [LM10].

Because our exceptional collections consist of line bundles, the corresponding quivers, Gram matrices etc., is easy to compute. Furthermore because our construction of the collections seems canonical, they behave well as in Step 1 above. We expect that there must be several applications of our choice of exceptional collections. In fact, as its application, we can prove the derived equivalence between some smooth toric 3-folds connected by a flop.

The structure of this paper is as follows: In §2, we give some basic definitions on the derived categories of coherent sheaves. In §3, we explain several notion on toric varieties and refer some useful results for smooth toric Fano 3-folds. We also explain how to determine the set $\mathfrak{D}_X$, following Thomsen. In §4, we actually determine it for several toric varieties. In §5, we show Bondal's conjecture for maximal smooth toric Fano 3-folds. In §6, we accomplish Step 1 above and give the proof of Theorem 1.1. In Theorem 6.3 we also obtain a similar result in the surface case to Theorem 1.1. In §7, using exceptional collections we constructed above, we show the derived equivalence between some smooth toric 3-folds connected by a flop.

Notation and conventions For a smooth variety X, we denote the bounded derived category of coherent sheaves on X by $D^b(X)$. *T-invariant* is an abbreviation of *torus invariant*. For objects $\mathcal{E}, \mathcal{F} \in D^b(X)$, we define

$$\mathrm{Hom}^i_X(\mathcal{E}, \mathcal{F}) := \mathrm{Hom}_{D^b(X)}(\mathcal{E}, \mathcal{F}[i]).$$

We work over $\mathbb{C}$ for simplicity.

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2 Generators of derived categories

In this section, we give several basic definitions on triangulated categories and derived categories of coherent sheaves.

Definition 2.1. Let $\mathcal{S} = \{\mathcal{S}_i\}$ be a set of objects in a triangulated category $\mathcal{D}$.

(i) We denote by $\langle \mathcal{S} \rangle$ the smallest triangulated subcategory of $\mathcal{D}$ containing all $\mathcal{S}_i$, closed under isomorphisms and direct summands. For a triangulated subcategory $\mathcal{C}$ of $\mathcal{D}$, we denote by $\mathcal{C}^{\perp}$ the full triangulated subcategory of $\mathcal{D}$ whose objects $\mathcal{F}$ satisfy the property $\mathrm{Hom}_{\mathcal{D}}(C, \mathcal{F}) = 0$ for all $C \in \mathcal{C}$.

(ii) We say that $\mathcal{S}$ *classically generates* $\mathcal{D}$ if $\langle \mathcal{S} \rangle = \mathcal{D}$. We also call $\mathcal{S}$ a *classical generator* of $\mathcal{D}$.

(iii) We say that $\mathcal{S}$ *generates* $\mathcal{D}$ if $\langle \mathcal{S} \rangle^{\perp} = 0$. We also call $\mathcal{S}$ a *generator* of $\mathcal{D}$.

Let X be a smooth complete variety over $\mathbb{C}$.

Definition 2.2. (i) An object $\mathcal{E} \in D^b(X)$ is called *exceptional* if it satisfies

$$\mathrm{Hom}^i_X(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(ii) An ordered set $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ of exceptional objects is called an *exceptional collection* if the following condition holds;

$$\mathrm{Hom}^i_X(\mathcal{E}_k, \mathcal{E}_j) = 0$$

for all $k > j$ and all i. When we say that a finite set $\mathcal{S}$ of objects is an exceptional collection, it means that $\mathcal{S}$ forms an exceptional collection in an appropriate order.

(iii) An exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ is called *strong* if

$$\mathrm{Hom}^i_X(\mathcal{E}_k, \mathcal{E}_j) = 0$$

for all k, j and $i \neq 0$.

(iv) An exceptional collection $(\mathcal{E}_1, \dots, \mathcal{E}_n)$ is called *full* if

$$\langle \mathcal{E}_1, \dots, \mathcal{E}_n \rangle = D^b(X).$$

Remark 2.3. If X has a full exceptional collection consisting of n exceptional objects, the rank of its K group $K(X)$ is n. Furthermore it is known that the rank of K group is the number of the maximal cones in the fan for smooth complete toric varieties. For 3-dimensional smooth complete toric varieties X, we can see $\operatorname{rank} K(X) = 2\rho(X) + 2$.

We have the following easy lemma. Because of it, the facts that the set $\mathfrak{D}_X$ forms a full strong exceptional collection and that $F_{m*}\mathcal{O}_X$ is a tilting generator for sufficiently large m are equivalent. See §3.3 for the definitions of $\mathfrak{D}_X$ and F_m.

Lemma 2.4. *(i) Let us consider a finite set of line bundles $\{\mathcal{L}_k\}$ satisfying $\mathcal{L}_i \ncong \mathcal{L}_j$ for $i \neq j$. Assume that the vector bundle $\mathcal{E} = \bigoplus \mathcal{L}_k$ is a* tilting generator *of $D^b(X)$, namely it satisfies the following conditions:*

(1) $\operatorname{Hom}^i_X(\mathcal{E}, \mathcal{E}) = 0$ for $i \neq 0$. Such an object $\mathcal{E}$ in $D^b(X)$ is called a tilting *object.*

(2) $\langle \mathcal{E} \rangle^{\perp} = 0$ in $D^b(X)$, that is, $\mathcal{E}$ is a generator *of $D^b(X)$.*

Then the set $\{\mathcal{L}_i\}$ forms a full strong exceptional collection.

(ii) Suppose that we have a full strong exceptional collection $\{\mathcal{L}_k\}$ on X. Then their direct sum $\bigoplus_k \mathcal{L}_k$ is a tilting generator of $D^b(X)$.

Proof. The most parts of the statements are direct consequences of the definitions. I explain only how to show the fullness in (i).

Note that the condition (1) implies that the set $\{\mathcal{L}_k\}$ is a strong exceptional collection. Then we have a semi-orthogonal decomposition (cf. [Hu06, page 25]) of $D^b(X)$ into $\langle \{\mathcal{L}_k\} \rangle^{\perp}$ and $\langle \{\mathcal{L}_k\} \rangle$. Since $\mathcal{E}$ is a generator, $\langle \{\mathcal{L}_k\} \rangle^{\perp} = 0$ which implies that the strong exceptional collection $\{\mathcal{L}_k\}$ is full. $\square$

3 Toric varieties

Throughout this section, we use the following notation. Let $N = \mathbb{Z}^n$ be a lattice of rank n and M its dual. A fan Δ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ consists of a finite number of rational strongly convex cones in $N_{\mathbb{R}}$, and it determines a toric variety $X = X(\Delta)$. We define $\mathcal{V}(\Delta)$ to be the set of primitive generators of 1-dimensional cones in Δ.

For a cone σ in Δ, put $R_\sigma = \mathbb{C}\big[\chi^{\boldsymbol{u}} \bigm| \boldsymbol{u} \in M \cap \sigma^\vee\big]$, where $\big\{\chi^{\boldsymbol{u}} \bigm| \boldsymbol{u} \in M \cap \sigma^\vee\big\}$ is a basis of $\mathbb{C}$-vector space R_σ. We can define an obvious multiplication on R as usual (cf. [Fu93, page 15]). Then the affine toric variety U_σ corresponds to σ is just $\operatorname{Spec} R_\sigma$, and the rational function field of the toric varieties X is just $\mathbb{C}\big[\chi^{\boldsymbol{u}} \bigm| \boldsymbol{u} \in M\big]$.

3.1 Double $\mathbb{Z}$-weight

According to [Od88], we introduce the notion of doubly $\mathbb{Z}$-weighted triangulations of a 2-sphere. For simplicity, we restrict to the case $n = 3$, namely $N = \mathbb{Z}^3$ below. We can obviously identify the set of half lines starting from the origin $\mathbf{0}$ of $N_{\mathbb{R}}$ with

$$S^2 := (N_{\mathbb{R}} \backslash \{\mathbf{0}\})/\mathbb{R}_{>0}.$$

Let

$$\pi\colon N_{\mathbb{R}} \backslash \{\mathbf{0}\} \to S^2$$

be the projection. We call $\pi(v)$ a *rational point* of S^2 corresponding to a primitive element $v \in N$, and v is called the *N-weight* of the rational point $\pi(v)$. For a cone $\sigma = \mathbb{R}_{\geq 0}\boldsymbol{v}_1 + \cdots + \mathbb{R}_{\geq 0}\boldsymbol{v}_s \in \Delta$ ($\boldsymbol{v}_i \in \mathcal{V}(\Delta)$), $\pi(\sigma\backslash\{\boldsymbol{0}\})$ is a convex spherical cell in S^2 with rational points $\pi(\boldsymbol{v}_1), \ldots, \pi(\boldsymbol{v}_s)$ as vertices. Thus for a fan Δ, we get a convex spherical cell decomposition

$$\left\{\pi(\sigma\backslash\{\boldsymbol{0}\} \mid \sigma \in \Delta\right\}$$

of $\pi(|\Delta|\backslash\{\boldsymbol{0}\})$.

Suppose that a fan Δ is complete and non-singular, which is equivalent to the condition that the corresponding toric variety $X = X(\Delta)$ is proper and smooth. Then we get a simplicial cell decomposition of S^2. Moreover, for each 3-dimensional cone $\sigma \in \Delta$, the corresponding spherical 2-simplex $\pi(\sigma\backslash\{\boldsymbol{0}\})$ has vertices whose N-weights $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ form a $\mathbb{Z}$-basis of N. For each 2-dimensional cone $\tau \in \Delta$, there are exactly two 3-dimensional cone $\sigma, \sigma' \in \Delta$ such that $\sigma \cap \sigma' = \tau$. In this case, the sets $\{\boldsymbol{v}, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ and $\{\boldsymbol{v}', \boldsymbol{v}_2, \boldsymbol{v}_3\}$ of N-weights for the vertices of $\pi(\sigma\backslash\{\boldsymbol{0}\})$ and $\pi(\sigma'\backslash\{\boldsymbol{0}\})$, respectively, are $\mathbb{Z}$-bases of N. Moreover we have

$$\boldsymbol{v} + \boldsymbol{v}' + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 = \boldsymbol{0}$$

for $\alpha_j \in \mathbb{Z}$ uniquely determined by τ. For

$$\rho = \mathbb{R}_{\geq 0}\boldsymbol{v}, \rho' = \mathbb{R}_{\geq 0}\boldsymbol{v}', \rho_j = \mathbb{R}_{\geq 0}\boldsymbol{v}_j \in \Delta \quad (j = 2, 3)\ ,$$

let D, D', D_j are the corresponding T-invariant divisors. Then it is known (cf. [Od88, page 81]) that we have

$$\alpha_j = (D_j \cdot D_2 \cdot D_3). \tag{1}$$

We then endow the edge $\pi(\boldsymbol{v}_2), \pi(\boldsymbol{v}_3)$ with the *double $\mathbb{Z}$-weight* α_2, α_3, where we place α_2 (resp. α_3) on the side of the vertex $\pi(\boldsymbol{v}_2)$ (resp. $\pi(\boldsymbol{v}_3)$) as in Figure 1. For simplicity, here and henceforth we always denote a rational point $\pi(\boldsymbol{v})$ by its N-weight $\boldsymbol{v}$ in figures of double $\mathbb{Z}$-weights.

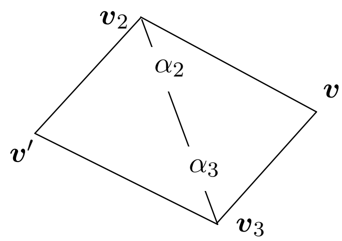

Figure 1: Double $\mathbb{Z}$-weight

We have normal bundle sequences;

$$\begin{aligned} 0 \to &\mathcal{N}_{C/D_2} \to \mathcal{N}_{C/X} \to \mathcal{N}_{D_2/X}|_C \to 0 \\ 0 \to &\mathcal{N}_{C/D_3} \to \mathcal{N}_{C/X} \to \mathcal{N}_{D_3/X}|_C \to 0, \end{aligned}$$

where $C \cong \mathbb{P}^1$ is the T-invariant curve corresponding to the cone τ. Then we know as above $\mathcal{N}_{C/D_j} \cong \mathcal{O}_{\mathbb{P}^1}(-\alpha_j)$ and $\mathcal{N}_{D_j/X}|_C \cong \mathcal{O}_{\mathbb{P}^1}(-\alpha_{j'})$, where $\{j, j'\} = \{2, 3\}$. Combining both sequences, we conclude that they split and so we have

$$\mathcal{N}_{C/X} \cong \mathcal{O}_{\mathbb{P}^1}(-\alpha_2) \oplus \mathcal{O}_{\mathbb{P}^1}(-\alpha_3). \tag{2}$$

We show in Figure 2 the change of double $\mathbb{Z}$-weights under the blowing-up along a T-invariant curve [Od88, page 90]. The segment attached to a oval corresponds to the curve, and the vertex with a dark gray small circle corresponds to the exceptional divisor. Figure 2 will be used to find the centers and the exceptional divisors of blowing-ups in Figure 4.

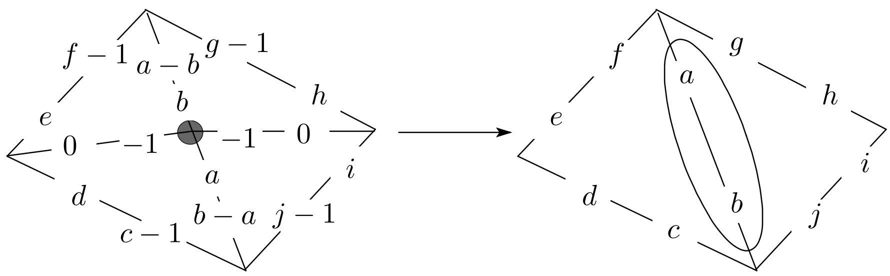

Figure 2: Change of double $\mathbb{Z}$-weights under the blowing-up

3.2 Toric Fano 3-folds

Smooth toric Fano 3-folds are classified as follows.

Theorem 3.1 ([Ba82, WW82])**.** *Up to isomorphism, there are* 18 *distinct Fano* 3*-folds. Among them, each of* (11), (12), (14), (15), (16) *and* (18) *below is obtained from one of the others by a finite succession of equivariant blowing-ups.*

(1) $\mathbb{P}^3$.

(2) $\mathbb{P}^2 \times \mathbb{P}^1$.

(3) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(1))$ *over* $Y = \mathbb{P}^2$.

(4) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(2))$ *over* $Y = \mathbb{P}^2$.

(5) *The* $\mathbb{P}^2$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y \oplus \mathcal{O}_Y(1))$ *over* $Y = \mathbb{P}^1$.

(6) $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

(7) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(f_1 + f_2))$ *over* $Y = \mathbb{P}^1 \times \mathbb{P}^1$, *where* f_1 *and* f_2 *are fibers of the two projections from* Y *to* $\mathbb{P}^1$.

(8) $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(f_1 - f_2))$ *in the notation of* (7).

(9) $\mathbb{P}^1 \times \Sigma_1$ *for the Hirzeburch surface* Σ_1.

(10) *The* $\mathbb{P}^1$*-bundle* $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(s + f))$ *over* $Y = \Sigma_1$, *where* f *is a fiber of the* $\mathbb{P}^1$*-bundle on* Σ_1 *and* s *is the minimal section with* $s^2 = -1$.

(13) $\mathbb{P}^1 \times Y_2$, *where* Y_2 *is the toric del Pezzo surface obtained from* $\mathbb{P}^2$ *by the equivariant blowing-up at two of the* T*-invariant points.*

(17) $\mathbb{P}^1 \times Y_3$, *where* Y_3 *is the toric del Pezzo surface obtained from* $\mathbb{P}^2$ *by the equivariant blowing-up at the three* T*-invariant points.*

Their birational relations are described in Figure 3. There are just three

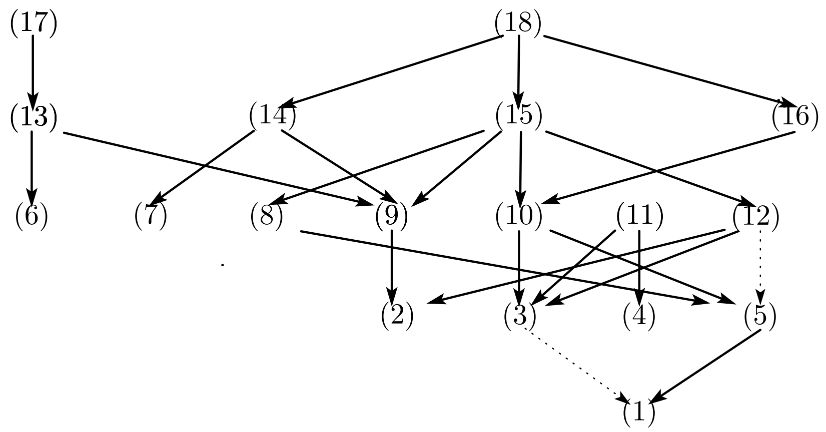

Figure 3: Every arrow means the equivariant blowing-up along a T-invariant curve and every dotted arrow means the equivariant blowing-up along a T-invariant point.

maximal Fano 3*-folds,* (11), (17) *and* (18), *with respect to birational relations.*

The corresponding eighteen doubly $\mathbb{Z}$-weighted triangulations of S^2 are given in Figure 4. A segment attached to an oval corresponds to the T-invariant smooth curve of the center of a blowing-up appearing in Figure 3. The number, like "(5)" in Figure 4(1), near the oval is the number of the Fano 3-fold obtained from the blowing-up.

For instance, the oval in Figure 4(1) means that if we blow up along the curve corresponding to the segment with the oval, we obtain the Fano 3-fold in (5). Of course, in this case, by symmetry we can choose any other segments, or any other T-invariant smooth curves, as the blowing-up center.

A dark gray small circle at a vertex corresponds to the exceptional T-invariant divisor of a blowing-down appearing in Figure 3. The number, like

"(1)" in Figure 4(5), near the small circle is the number of the Fano 3-fold obtained from the blowing-down.

For instance, the small circle in Figure 4(5) means that if we blow down the T-invariant divisor corresponding to the vertex with the small circle, we obtain the Fano 3-fold in (1).

In Figure 4, we do not indicate the point of a blowing-up center, or the exceptional divisor of a blowing-down to a point, since we do not need it afterwards.

3.3 Frobenius push-forward

In this subsection, we explain how to compute the direct summands of Frobenius push-forward of line bundles on smooth complete toric varieties, following Thomsen [Th00].

Fix a positive integer m and we define a new lattice N' as $N' := \frac{1}{m}N$ and denote its dual by M', We consider the natural inclusion $f_m : N \hookrightarrow N'$, which sends a cone in $N_{\mathbb{R}}$ to itself. Thus f_m induces the finite surjective toric morphism $F_m : X(\Delta) \to X(\Delta)$. We call a map F_m *Frobenius map.* It is also called a *multiplication map.* We put

$$\mathcal{V}(\Delta) = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l\},$$

and D_i to be the T-invariant divisor corresponding to the 1-dimensional cone generated by $\boldsymbol{v}_i$. Henceforth, without otherwise specified, we always assume that Δ is a complete smooth fan, i.e. $X = X(\Delta)$ is a smooth complete toric variety . We put $A = {}^t(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_l) \in M(l, n)$.

If $D = \sum_{j=1}^{l} b_j D_j$ is a $\mathbb{Q}$-divisor, we define

$$\lceil D \rceil := \sum_{j=1}^{l} \lceil b_i \rceil D_j,$$

where for any real number x, $\lceil x \rceil$ is the integer defined by $x \leq \lceil x \rceil < x + 1$. Similarly, we define

$$\lfloor D \rfloor := \sum_{j=1}^{l} \lfloor b_j \rfloor D_j,$$

where for every x, $\lfloor x \rfloor$ is the integer defined by $x - 1 < \lfloor x \rfloor \leq x$. K_X denotes the canonical divisor $-\sum_{j=1}^{l} D_j$ so that $\omega_X = \mathcal{O}_X(K_X)$.

For every cone $\sigma \in \Delta$, we denote by U_σ the open subset of X corresponding to σ. For a maximal cone

$$\sigma = \langle \boldsymbol{v}_{i_1}, \ldots, \boldsymbol{v}_{i_n} \rangle \quad (i_1 < \cdots < i_n)$$

and a matrix

$$B = {}^t(\boldsymbol{b}_1, \ldots, \boldsymbol{b}_l) \in M(l, m) \quad (m \geq 1),$$

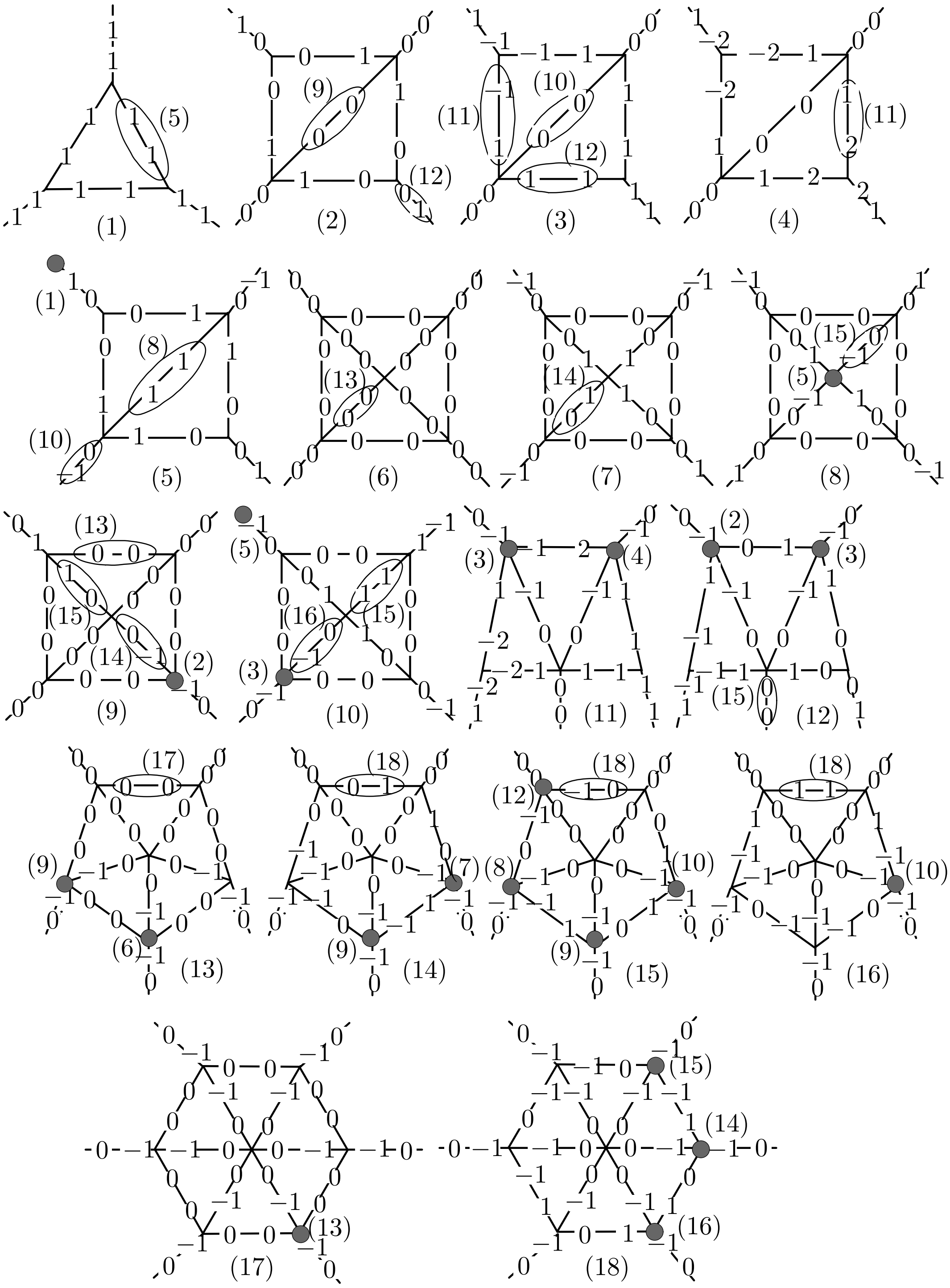

Figure 4: Doubly $\mathbb{Z}$-weighted triangulations of S^2 [Od88, page 91]

we define

$$B_\sigma = {}^t\!(\boldsymbol{b}_{i_1}, \ldots, \boldsymbol{b}_{i_n}) \in M(n, m).$$

Note that for a maximal cone σ, A_σ belongs to $GL(n, \mathbb{Z})$, since X is a smooth toric variety.

Put

$$P_m^p = \{ {}^t\!(u_1, \ldots, u_p) \in \mathbb{Z}^p \mid 0 \leqq u_i < m\}$$

for a positive integer p. For $\boldsymbol{u} \in P_m^n$, $\boldsymbol{w} = {}^t\!(w_1, \ldots, w_l) \in \mathbb{Z}^l$ and a maximal cone $\sigma \in \Delta$, define $\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \in \mathbb{Z}^l, \boldsymbol{r}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \in P_m^l$ and $q_i^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \in \mathbb{Z}$ as

$$AA_\sigma^{-1}(\boldsymbol{u} - \boldsymbol{w}_\sigma) + \boldsymbol{w} = m\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) + \boldsymbol{r}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) \tag{3}$$

and

$$\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) = {}^t\!(q_1^m(\boldsymbol{u}, \boldsymbol{w}, \sigma), \ldots, q_l^m(\boldsymbol{u}, \boldsymbol{w}, \sigma)).$$

Define

$$D_{\boldsymbol{u},\boldsymbol{w},\sigma} (= D^m_{\boldsymbol{u},\boldsymbol{w},\sigma}) := \sum q_i^m(\boldsymbol{u}, \boldsymbol{w}, \sigma) D_i.$$

Remark 3.2. (i) Suppose that

$$\boldsymbol{a} - \boldsymbol{b} = A\boldsymbol{u}$$

for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}^l$ and $\boldsymbol{u} \in \mathbb{Z}^n$. Then we know that

$$\sum_{i=1}^{l} a_i D_i - \sum_{i=1}^{l} b_i D_i = \operatorname{div} \chi^{\boldsymbol{u}}.$$

In particular the divisors $\sum_{i=1}^l a_i D_i$ and $\sum_{i=1}^l b_i D_i$ are linearly equivalent.

(ii) We have $\operatorname{div} \chi^{A_\sigma^{-1}\boldsymbol{q}}|_{U_\sigma} = \sum_{i=1}^n q_i D_i|_{U_\sigma}$ for any $\boldsymbol{q} = {}^t\!(q_1, \ldots, q_n) \in \mathbb{Z}^n$.

Example 3.3. Put $R = R_\sigma$ for an n-dimensional non-singular strongly convex rational cone σ in N. For a smooth affine toric variety $U = \operatorname{Spec} R$, the multiplication map F_m induces a $\mathbb{C}$-algebra map

$$F_m^\#\colon R \to R \quad \chi^{\boldsymbol{u}} \mapsto \chi^{m\boldsymbol{u}}.$$

When we regard a quotient field K of R as an R-module via the map $F_m^\#$, we denote it by $F_{m*}K$. For a sub-R-module L of K, we also define a sub-R-module $F_{m*}L$ of $F_{m*}K$, which is just L as an abelian group.

Then the module $F_{m*}(R\chi^{-A_\sigma^{-1}\boldsymbol{w}})$ for some $\boldsymbol{w} \in \mathbb{Z}^n$ is freely generated by the set

$$\{\chi^{A_\sigma^{-1}(\boldsymbol{u}-\boldsymbol{w})} \mid \boldsymbol{u} \in P_m^n\},$$

namely, we have an isomorphism

$$F_{m*}(R\chi^{-A_\sigma^{-1}\boldsymbol{w}}) \cong \bigoplus_{\boldsymbol{u}\in P_m^n} R\chi^{A_\sigma^{-1}(\boldsymbol{u}-\boldsymbol{w})},$$

since we also have the following description;

$$R = k\bigl[\chi^{A_\sigma^{-1}\boldsymbol{e}_i} \bigm| i = 1,\dots,n\bigr],$$

where $\bigl\{\boldsymbol{e}_1,\dots,\boldsymbol{e}_n\bigr\}$ is the standard basis of M.

The isomorphism on an affine piece in Example 3.3 can be globalized as follows in (ii).

Lemma 3.4. *Fix a vector* $\boldsymbol{w} = {}^t(w_1,\dots,w_l) \in \mathbb{Z}^l$ *and a maximal cone* $\sigma \in \Delta$.

(i) [Th00] The vector bundle

$$\bigoplus_{\boldsymbol{u}\in P_m^n} \mathcal{O}_X(D_{\boldsymbol{u},\boldsymbol{w},\sigma})$$

does not depend on the choice of a maximal cone σ.

(ii) [Th00] We have

$$\bigoplus_{\boldsymbol{u}\in P_m^n} \mathcal{O}_X(D_{\boldsymbol{u},\boldsymbol{w},\sigma}) \cong F_{m*}\mathcal{O}_X(\sum w_i D_i).$$

(iii) For a line bundle $\mathcal{L} \in \operatorname{Pic} X$*, we have*

$$(F_{m*}\mathcal{L})^\vee \cong F_{m*}(\mathcal{L}^\vee \otimes \omega_X^{1-m}) \cong F_{m*}(\mathcal{L}^\vee \otimes \omega_X) \otimes \omega_X^{-1}.$$

Proof. (iii) The second isomorphism follows from the projection formula. The first one is a direct consequence of the Grothendieck–Verdier duality (cf. [Hu06, p.86]), but we give another proof by the use of the above result. Put $\mathcal{L} = \mathcal{O}_X(\sum w_i D_i)$. We have $\mathcal{L}^\vee \otimes \omega_X^{1-m} = \mathcal{O}_X(\sum(m-1-w_i)D_i)$. Put $\boldsymbol{u}' = (m-1)\,{}^t(1,1,\dots,1)$. Then, for all $\boldsymbol{u} \in P_m^n$, we can see

$$\begin{aligned} q_i(\boldsymbol{u}'_{\sigma_0} - \boldsymbol{u}, \boldsymbol{u}' - \boldsymbol{w}, \sigma_0) =& \lfloor \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u}'_{\sigma_0} - \boldsymbol{u} - \boldsymbol{u}'_{\sigma_0} + \boldsymbol{w}_{\sigma_0}) - w_i + m - 1}{m} \rfloor \\ =& \lfloor -\frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + w_i + 1}{m} \rfloor + 1 \\ =& -\lceil \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + w_i}{m} + \frac{1}{m} \rceil + 1 \\ =& -q_i(\boldsymbol{u}, \boldsymbol{w}, \sigma_0). \end{aligned}$$

The last equality holds because, in general, the equality $\lceil \frac{k}{m} + \frac{1}{m} \rceil - \lfloor \frac{k}{m} \rfloor = 1$ is true for $k \in \mathbb{Z}$. This gives the first isomorphism by (iii). $\square$

Below for simplicity, we often identify two isomorphic line bundles. For a T-invariant divisor D and an integer $m > 0$, we define sets of (isomorphism classes of) line bundles;

$$\mathfrak{D}(D)_m := \{\mathcal{L} \in \operatorname{Pic} X \bigm| \mathcal{L} \text{ is a direct summand of } F_{m*}\mathcal{O}_X(D)\ \}$$

and

$$\mathfrak{D}(D) := \cup_{m>0}\mathfrak{D}(D)_m.$$

Convention

(i) We may assume that $\boldsymbol{v}_1, \dots, \boldsymbol{v}_n$ forms a standard basis of $\mathbb{Z}^n$ and put $\sigma_0 = \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_n \rangle$. We often omit σ_0 in the notation as $\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w})(:= \boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}, \sigma_0))$, $D_{\boldsymbol{u},\boldsymbol{w}} = D_{\boldsymbol{u},\boldsymbol{w},\sigma_0}$ and so on.

(ii) For a zero divisor $D = 0$ or a zero vector $\boldsymbol{w} = \boldsymbol{0}$, we simply denote $\mathfrak{D}(0)$ by $\mathfrak{D}$ (or $\mathfrak{D}_X$ if we need to specify the base variety X) and $\boldsymbol{q}^m(\boldsymbol{u})(= \boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{0}))$. (In fact, as a consequence of Lemma 3.5(i) and (ii), we have $\mathfrak{D} = \mathfrak{D}(0)_m$ for a sufficiently divisible integer m.)

We take σ in (3) to be σ_0, then (3) becomes the following simpler form

$$A(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + \boldsymbol{w} = m\boldsymbol{q}^m(\boldsymbol{u}, \boldsymbol{w}) + \boldsymbol{r}^m(\boldsymbol{u}, \boldsymbol{w}), \tag{4}$$

and hence we have

$$q_i^m(\boldsymbol{u}, \boldsymbol{w}) = \lfloor \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - \boldsymbol{w}_{\sigma_0}) + w_i}{m} \rfloor. \tag{5}$$

Lemma 3.5. *Fix a T-invariant divisor $D = \sum w_i D_i$ and put $\boldsymbol{w} = {}^t(w_1, \dots, w_l)$.*

(i) [Th00] The set $\mathfrak{D}(D)$ is finite.

(ii) Put $D' := \sum w_i' D_i$, where

$$w_i' := \begin{cases} 0 & \text{for } i \text{ with } w_i \geq 0 \\ -1 & \text{for } i \text{ with } w_i < 0. \end{cases}$$

Take $m > 0$ satisfying $-1 \leq \frac{w_i}{m} < 1$ for any i. Then $\mathcal{O}_X(D') \in \mathfrak{D}(D)_m$. Furthermore we have $\mathfrak{D}(D') \subset \mathfrak{D}(D)_m$ for sufficiently divisible integers $m > 0$.

(iii) We have $\mathfrak{D}(D)_m \subset \mathfrak{D}(lD)_{lm}$ for any $l, m \in \mathbb{Z}_{>0}$.

Proof. (i) Since the set $\left\{ \frac{{}^t\boldsymbol{v}_i\boldsymbol{u}}{m} \bigm| \boldsymbol{u} \in P_m^n, m \in \mathbb{Z}_{>0} \right\}$ is bounded and $\frac{{}^t\boldsymbol{v}_i\boldsymbol{w}_{\sigma_0} - w_i}{m} \to 0$ as $m \to \infty$, the set of integers $\left\{ q_i^m(\boldsymbol{u}, \boldsymbol{w}) \bigm| \boldsymbol{u} \in P_m^n, m \in \mathbb{Z}_{>0} \right\}$ is finite. Consequently, so is the set $\mathfrak{D}(D)$.

(ii) For any $\boldsymbol{w}_{\sigma_0} \in \mathbb{Z}^n$, there is a vector $\boldsymbol{u}' \in \mathbb{Z}^n$ such that $m\boldsymbol{u}' + \boldsymbol{w}_{\sigma_0} \in P_m^n$. Put $\boldsymbol{u} := m\boldsymbol{u}' + \boldsymbol{w}_{\sigma_0}$. Then we can see

$$\begin{aligned} q_i^m(\boldsymbol{u}, \boldsymbol{w}) =& \lfloor \frac{{}^t\boldsymbol{v}_i(\boldsymbol{u} - m\boldsymbol{u}' - \boldsymbol{w}_{\sigma_0}) + w_i}{m} \rfloor + \frac{m\,{}^t\boldsymbol{v}_i\boldsymbol{u}'}{m} \\ =& \lfloor \frac{w_i}{m} \rfloor + {}^t\boldsymbol{v}_i\boldsymbol{u}' = w_i' + {}^t\boldsymbol{v}_i\boldsymbol{u}', \end{aligned}$$

which means the divisor $D_{\boldsymbol{u},\boldsymbol{w}}$ is linearly equivalent to D'. Thus $\mathcal{O}_X(D') \in \mathfrak{D}(D)_m$.

Take an element $\mathcal{L} \in \mathfrak{D}(D')$ and suppose that $\mathcal{L} \in \mathfrak{D}(D)_m$ for some m satisfying $-1 \leq \frac{w_i}{m} < 1$ for any i. Then $\mathcal{L} \in \mathfrak{D}(D)_{km}$ for all $k > 0$, since $F_{km*}\mathcal{O}_X(D) = F_{m*}F_{k*}\mathcal{O}_X(D)$. This gives the last assertion.

(iii) By definition, we have $q_i^m(\boldsymbol{u}, \boldsymbol{w}) = q_i^{lm}(l\boldsymbol{u}, l\boldsymbol{w})$, which implies the conclusion.

□

My optimistic conjecture is as follows;

Conjecture 3.6. *Let X be a smooth complete toric variety. Then $F_{m*}\mathcal{O}_X(D)$ is a classical generator of $D^b(X)$ for any T-invariant divisors D and a sufficiently large integer m.*

In order to prove Conjecture 3.6, by Lemma 3.5(ii) it is essential to show it for T-invariant divisors $D = \sum w_i D_i$ with $w_i = 0$ or -1.

Remark 3.7. Bondal announced in [Bo06] that Conjecture 3.6 is true for the case $D = 0$. Although the proof is not available so far, several people have already used this statement (cf. [BT10, CM10, CM, DLM10]). In this article, we refer this statement as *Bondal's conjecture.*

Lemma 3.5 will not be used afterwards, but an idea to solve Bondal's conjecture in §4.3 and 4.4 comes from it.

The following is sometimes powerful when we show that $F_{m*}\mathcal{O}_X$ is a tilting object.

Lemma 3.8. *Take a line bundle $\mathcal{L}$ on X.*

(i) If $\mathcal{L}^{-1}$ is nef, then we have $\mathrm{Ext}_X^i(\mathcal{L}, F_{m*}\mathcal{O}_X) = 0$ for $i > 0$.

(ii) [Sa09] If $\mathcal{L} \otimes \omega_X^{-1}$ is ample, then we have $\mathrm{Ext}_X^i(F_{m*}\mathcal{O}_X, \mathcal{L}) = 0$ for $i > 0$.

Proof. (i) By adjunction, we have $\mathrm{Ext}_X^i(\mathcal{L}, F_{m*}\mathcal{O}_X) = H^i(X, \mathcal{L}^{-m})$. But the last term vanishes for $i > 0$, since X is toric.

(ii) We have $\mathrm{Ext}_X^i(F_{m*}\mathcal{O}_X, \mathcal{L}) = H^i(X, F_m^*(\mathcal{L} \otimes \omega_X^{-1}) \otimes \omega_X)$, which vanishes by the Kodaira vanishing theorem. □

4 Examples

In this section, we determine the set $\mathfrak{D}$ for various smooth toric varieties.

4.1 Hirzeburch surfaces

Let us consider a Hirzeburch surface $X = \Sigma_d$ over $\mathbb{P}^1$. Define

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ d \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \in \mathbb{Z}^2.$$

Then we know that D_1 and D_3 are irreducible fibers of a $\mathbb{P}^1$-bundle on X, and D_2 and D_4 are sections with $(D_2)^2 = -d$ and $(D_4)^2 = d$. Note that $D_1 \sim D_3$ and $D_2 \sim D_4 - dD_3$.

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix} \in P^2_m$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{-x+dy}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lfloor \frac{-x+dy}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \end{pmatrix}.$$

Therefore $D_{\boldsymbol{u}} = \lfloor \frac{-x+dy}{m} \rfloor D_3 + \lfloor \frac{-y}{m} \rfloor D_4$. To determine $\mathfrak{D}$, we may assume that $m \gg d$ by Lemma 3.5. Then we obtain

$$\mathfrak{D} = \left\{ \mathcal{O}_X(-D_3 - D_4), \mathcal{O}_X(-D_4), \mathcal{O}_X(-D_3), \mathcal{O}_X, \mathcal{O}_X(iD_3 - D_4) \;\middle|\; 1 \leqq i \leqq d-1 \right\},$$

and hence

$$\begin{aligned} \mathfrak{D}(K_X) = \{ & \mathcal{O}_X((d-1)D_3 - D_4), \mathcal{O}_X((d-2)D_3 - D_4), \mathcal{O}_X((d-1)D_3 - 2D_4), \\ & \mathcal{O}_X((d-2)D_3 - 2D_4) = \omega_X, \mathcal{O}_X((d-2-i)D_3 - D_4) \mid 1 \leqq i \leqq d-1 \} \end{aligned}$$

by Lemma 3.4(iii) or a direct computation by the use of (4) for $\boldsymbol{w} = {}^t(-1, \ldots, -1)$.

Now we have

$$\begin{aligned} \mathrm{Hom}^1_X(\mathcal{O}_X(iD_3 - D_4), \mathcal{O}_X(-D_3)) &= H^1(X, \mathcal{O}_X((-i-1)D_3 + D_4)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d-i-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-i-1)), \end{aligned}$$

which is non-zero for $d \geq 2$ and $d - 1 \geq i \geq 1$. Hence we see that $F_{m*}\mathcal{O}_X$ is not a tilting object for $d \geq 2$. On the other hand, it is known that

$$\mathcal{O}_X(-D_3 - D_4) \oplus \mathcal{O}_X(-D_4) \oplus \mathcal{O}_X(-D_3) \oplus \mathcal{O}_X$$

is a tilting generator for all d (cf. [Ki]).

4.2 Maximal toric del Pezzo surface

Let us consider the toric surface $X = Y_3$ which is obtained by the blow up of $\mathbb{P}^2$ at the three T-invariant points. Namely, X is the *maximal* toric del Pezzo surface with respect to birational relations. We put

$$\begin{aligned} \boldsymbol{v}_1 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \\ \boldsymbol{v}_4 &= \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \in \mathbb{Z}^2. \end{aligned}$$

Then we know that D_2, D_4, D_6 are exceptional divisors of $X \to \mathbb{P}^2$. Note that $D_1 + D_6 \sim D_3 + D_4$ and $D_2 + D_3 \sim D_5 + D_6$.

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \end{pmatrix} \in P_m^2$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{-x+y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \lfloor \frac{-x+y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \end{pmatrix}.$$

Then we obtain

$$\begin{aligned} \mathfrak{D} = \{ &\mathcal{O}_X(-D_5 - D_6), \mathcal{O}_X(-D_3 - D_4), \mathcal{O}_X(-D_4 - D_5), \\ &\mathcal{O}_X(-D_3 - D_4 - D_5), \mathcal{O}_X(-D_4 - D_5 - D_6), \mathcal{O}_X \}. \end{aligned}$$

These are dual of the line bundles which appear in a full strong exceptional collections on X in [Ki]. In particular, $\mathfrak{D}$ forms a full strong exceptional collection.

4.3 Fano 3-fold in (11)

Take the Fano 3-fold X in (11). Put

$$\begin{aligned} \boldsymbol{v}_1 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \\ \boldsymbol{v}_4 &= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \in \mathbb{Z}^2, \end{aligned}$$

as in Figure 5.

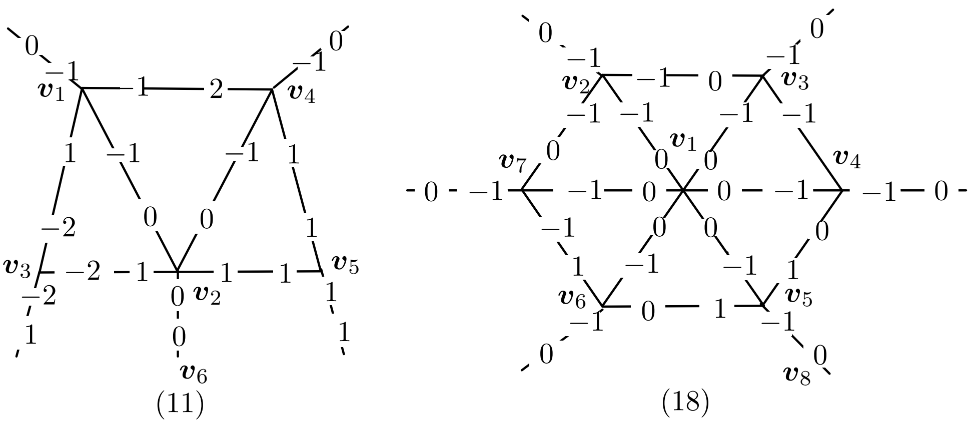

Figure 5: Fano 3-folds in (11) and (18)

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_m^3$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{z}{m} \rfloor \\ \lfloor \frac{x-z}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{-x-y+2z}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{x-z}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{-x-y+2z}{m} \rfloor \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}\mathfrak{D} = \big\{&\mathcal{O}_X, \mathcal{O}_X(-D_6), \mathcal{O}_X(-2D_6), \mathcal{O}_X(-D_5), \mathcal{O}_X(-D_5 - D_6), \mathcal{O}_X(-D_5 - 2D_6),\\ &\mathcal{O}_X(-D_4 - D_5 - D_6), \mathcal{O}_X(-D_4 - D_5), \mathcal{O}_X(-D_4 - D_5 + D_6)\big\}.\end{aligned}$$

Then by the equation (1) we can read from Figure 5 that for all $\mathcal{L} \in \mathfrak{D}$ except $\mathcal{O}_X(-D_4 - D_5 + D_6)$, $\mathcal{L}^{-1}$ is nef, hence Lemma 3.8 implies that $\operatorname{Ext}^i_X(\mathcal{L}, F_{m*}\mathcal{O}_X) = 0$ for $i > 0$. Put

$$\mathfrak{D}_{\text{nef}} := \mathfrak{D} \backslash \{\mathcal{O}_X(-D_4 - D_5 + D_6)\}.$$

We shall prove in §5.2 that $\langle \mathfrak{D}_{\text{nef}} \rangle = D^b(X)$. Consequently the set $\mathfrak{D}_{\text{nef}}$ becomes a full strong exceptional collection. We also know that

$$\operatorname{Ext}^i_X(\mathcal{O}_X(-D_4 - D_5 + D_6), F_{m*}\mathcal{O}_X) \neq 0$$

for some $i > 0$, since, otherwise, $\mathfrak{D}$ becomes a full strong exceptional collection. This contradicts with $\operatorname{rank} K(X) = 8 \neq 9 = |\mathfrak{D}|$.

4.4 Fano 3-fold in (18)

Take the Fano 3-fold in (18). Put

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix},$$
$$\boldsymbol{v}_5 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_7 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \boldsymbol{v}_8 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \in \mathbb{Z}^2,$$

as in Figure 5. For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P^3_m$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{z}{m} \rfloor \\ \lfloor \frac{-y+z}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{y-z}{m} \rfloor \\ \lfloor \frac{-x+y}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{-y+z}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{y-z}{m} \rfloor \\ \lfloor \frac{-x+y}{m} \rfloor \end{pmatrix}.$$

Therefore we have

$$\begin{aligned} \mathfrak{D} = \big\{ & \mathcal{O}_X(-iD_8), \mathcal{O}_X(-D_6 - D_7 - iD_8), \mathcal{O}_X(-D_4 - D_5 - iD_8), \\ & \mathcal{O}_X(-D_5 - D_6 - D_7 - iD_8), \mathcal{O}_X(-D_5 - D_6 - iD_8), \\ & \mathcal{O}_X(-D_4 - D_5 - D_6 - iD_8) \bigm| i = 0, 1 \big\}. \end{aligned}$$

By the equation (1), we can read from Figure 5 that that $\mathcal{L}^{-1}$ is nef for all $\mathcal{L} \in \mathfrak{D}$. Hence by Lemma 3.8 implies that $\operatorname{Ext}^i_X(F_{m*}\mathcal{O}_X, F_{m*}\mathcal{O}_X) = 0$ for all $m \gg 0, i > 0$.

4.5 Fano 3-fold in (8)

Take the Fano 3-fold X in (8). Put

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$
$$\boldsymbol{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \boldsymbol{v}_6 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{Z}^2.$$

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_m^3$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{z}{m} \rfloor \\ \lfloor \frac{-x-z}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{-x-z}{m} \rfloor \\ \lfloor \frac{x-y}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \end{pmatrix}.$$

Therefore we have

$$\begin{aligned}\mathfrak{D} = \big\{ &\mathcal{O}_X, \mathcal{O}_X(-D_5), \mathcal{O}_X(-D_4), \mathcal{O}_X(-D_4 - D_5), \mathcal{O}_X(-D_4 - D_6), \\ &\mathcal{O}_X(-D_4 - D_5 - D_6), \mathcal{O}_X(-2D_4 - D_6), \mathcal{O}_X(-2D_4 - D_5 - D_6) \big\}.\end{aligned}$$

For all of line bundles $\mathcal{L} \in \mathfrak{D}$, we can see that $\mathcal{L}^{-1}$ is nef by a similar method to one above, hence Lemma 3.8 implies that $\mathrm{Ext}^i_X(F_{m*}\mathcal{O}_X, F_{m*}\mathcal{O}_X) = 0$ for $i > 0$. This result contradicts the result in [Sa09, page 32].

4.6 Fano 4-fold

We consider the Fano 4-fold $X = V^4$ corresponding to the polytope

$$Conv \langle \boldsymbol{v}_1, \ldots, \boldsymbol{v}_{10} \rangle,$$

where

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ldots, \boldsymbol{v}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ldots, \boldsymbol{v}_8 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$
$$\boldsymbol{v}_9 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \boldsymbol{v}_{10} = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

Note that X is a maximal Fano 4-folds with $\rho(X) = 6$ and $\operatorname{rank} K(X) = 30$.

For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in P_m^4$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}) = \begin{pmatrix} \lfloor \frac{x}{m} \rfloor \\ \lfloor \frac{y}{m} \rfloor \\ \lfloor \frac{z}{m} \rfloor \\ \lfloor \frac{w}{m} \rfloor \\ \lfloor \frac{-x}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{-w}{m} \rfloor \\ \lfloor \frac{x+y+z+w}{m} \rfloor \\ \lfloor \frac{-x-y-z-w}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \lfloor \frac{-x}{m} \rfloor \\ \lfloor \frac{-y}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{-w}{m} \rfloor \\ \lfloor \frac{x+y+z+w}{m} \rfloor \\ \lfloor \frac{-x-y-z-w}{m} \rfloor \end{pmatrix}.$$

Hence the set $\mathfrak{D}_X$ consists of the following 50(=1+4+6+12+4+16+1+6) line bundles;

$\mathcal{O}_X, \mathcal{O}_X(-D_i - D_{10})$ (4 line bundles),
$\mathcal{O}_X(-D_i - D_j - D_{10})$ (6 line bundles),
$\mathcal{O}_X(-D_i - D_j + a_1 D_9 - b_1 D_{10})$ (12 line bundles),
$\mathcal{O}_X(-D_i - D_j - D_k - D_{10})$ (4 line bundles),
$\mathcal{O}_X(-D_i - D_j - D_k + a_2 D_9 - b_2 D_{10})$ (16 line bundles),
$\mathcal{O}_X(-D_5 - D_6 - D_7 - D_8 - D_{10})$,
$\mathcal{O}_X(-D_5 - D_6 - D_7 - D_8 + a_3 D_9 - b_3 D_{10})$ (6 line bundles),

where $b_h - a_h = 0$ or 1, $a_h = 1, \ldots, h$ and $i, j, k \in \{5, 6, 7, 8\}$ with $(i-j)(j-k)(i-k) \neq 0$.

As you can see above, the situation becomes worse in the 4-dimensional case. In the higher dimensional cases it becomes much worse. It is known by [LM10] that there is a smooth toric Fano variety X such that we cannot choose full strong exceptional collections from the set $\mathfrak{D}_X$.

5 Exceptional collections on maximal toric Fano 3-folds

In this section, we show Bondal's conjecture for maximal smooth toric Fano 3-folds. Combining this with the results in §4, we see that $\mathfrak{D}_X$ (respectively, $\mathfrak{D}_{\mathrm{nef}}$) is a full strong exceptional collection in the cases Fano 3-folds in (17) and (18) (respectively, (11)).

Lemma 5.1. *Let $f\colon X \to Y$ be a birational contraction between smooth projective varieties. Suppose that an object $\mathcal{E}$ is a generator of $D^b(X)$. Then $\mathbb{R}f_*\mathcal{E}$ is also a generator of $D^b(Y)$.*

Proof. Put $\mathbb{R}\mathrm{Hom}_Y(\mathbb{R}f_*\mathcal{E},\mathcal{F}) = 0$ for some $\mathcal{F} \in D^b(Y)$. Then by the adjointness $\mathbb{R}f_* \dashv f^!$, we obtain $\omega_X \otimes \mathbb{L}f^*(\mathcal{F} \otimes \omega_Y^{-1}) = f^!\mathcal{F} = 0$, which implies that $\mathcal{F} = 0$. □

For the toric case, by $\mathbb{R}f_*\mathcal{O}_X = \mathcal{O}_Y$ and the commutativity $F_m^Y \circ f = f \circ F_m^X$, we have $\mathbb{R}f_*F_{m*}^X\mathcal{O}_X = F_{m*}^Y\mathcal{O}_Y$.

5.1 Exceptional collection on the Fano 3-fold in (17)

The following must be well-known.

Lemma 5.2. *Let Y and Z be smooth projective varieties. Suppose that $\mathcal{E}$ and $\mathcal{F}$ are tilting generators of $D^b(Y)$ and $D^b(Z)$ respectively. Then $\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}$ is also a tilting generator of $D^b(Y \times Z)$.*

Proof. We can check

$$\mathbb{R}\Gamma(Y \times Z, \mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}) = \mathbb{R}\Gamma(Y,\mathcal{E}) \overset{\mathbb{L}}{\otimes} \mathbb{R}\Gamma(Z,\mathcal{F}).$$

Hence we have

$$\mathrm{Hom}^i_{Y\times Z}(\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}, \mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}) \cong \bigoplus_{j+k=i} \mathrm{Hom}^j_Y(\mathcal{E},\mathcal{E}) \otimes \mathrm{Hom}^k_Z(\mathcal{F},\mathcal{F}),$$

which implies that $\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}$ is tilting. The fact $\mathcal{E} \overset{\mathbb{L}}{\boxtimes} \mathcal{F}$ is a generator directly follows from [BV03, Lemma 3.4.1].

□

For the toric case, we know that $F_{m*}^{Y\times Z}\mathcal{O}_{Y\times Z} \cong F_{m*}^Y\mathcal{O}_Y \boxtimes F_{m*}^Z\mathcal{O}_Z$. By §4.2, we see that $F_{m*}\mathcal{O}_{Y_3}$ is a tilting generator on the maximal toric del Pezzo surface Y_3 and $m \gg 0$. Because the Fano 3-fold X in (17) is the product of Y_3 and a projective line $\mathbb{P}^1$ it follows that $\mathfrak{D}_X$ is a full strong exceptional collection. Here we leave to readers the proof of the fact that $\mathfrak{D}_{\mathbb{P}^1}$ is a full strong exceptional collection on $\mathbb{P}^1$.

5.2 Exceptional collection on Fano 3-fold in (11)

Take the Fano 3-fold X in (11). We use the same notation as in §4.3. First we determine the set $\mathfrak{D}(\omega_X^{-3})_m$ for sufficiently large m. For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in P_m^3$

and $\boldsymbol{w} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \in \mathbb{Z}^6$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{x-z+3}{m} \rfloor \\ \lfloor \frac{-z}{m} \rfloor \\ \lfloor \frac{-x-y+2z+3}{m} \rfloor \end{pmatrix}.$$

Thus we have

$$q_4^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } x+3 \geq z+m \\ 0 & \text{if } z+m > x+3 \geq z \\ -1 & \text{if } z > x+3 \end{cases}$$

$$q_5^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 0 & \text{if } z = 0 \\ -1 & \text{if } z > 0 \end{cases}$$

$$q_6^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 2 & \text{if } 2z+3 \geq x+y+2m \\ 1 & \text{if } x+y+2m > 2z+3 \geq x+y+m \\ 0 & \text{if } x+y+m > 2z+3 \geq x+y \\ -1 & \text{if } x+y > 2z+3 \geq x+y-m \\ -2 & \text{if } x+y-m > 2z+3. \end{cases}$$

By tedious computation, we can see $\mathfrak{D}(\omega_X^{-3})_m = \mathfrak{D} \cup \mathfrak{D}'$, where

$$\mathfrak{D}' = \big\{\mathcal{O}_X(D_4 - iD_5 - jD_6), \mathcal{O}_X(-D_5 + D_6), \\ \mathcal{O}_X(-D_4 - D_5 + 2D_6) \bigm| i = 1, 0 \text{ and } j = 1, 2\big\}.$$

Note that there are linear equivalences;

$$D_1 + D_4 \sim D_6, \quad D_2 \sim D_6, \quad D_3 + 2D_6 \sim D_4 + D_5. \tag{6}$$

We shall check that $\mathcal{L} \in \langle \mathfrak{D}_{\text{nef}} \rangle$ for $\mathcal{L} = \mathcal{O}_X(-D_4 - D_5 + D_6)$ or all $\mathcal{L} \in \mathfrak{D}'$ below.

Since $D_1 \cap D_2 \cap D_6 = \emptyset$, we have an exact sequence

$$\begin{aligned} 0 \to & \mathcal{O}_X(-D_1 - D_2 - D_6) \\ \to & \mathcal{O}_X(-D_1 - D_2) \oplus \mathcal{O}_X(-D_2 - D_6) \oplus \mathcal{O}_X(-D_1 - D_6) \\ \to & \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_2) \oplus \mathcal{O}_X(-D_6) \to \mathcal{O}_X \to 0. \end{aligned}$$

Combining this with (6), we have an exact sequence

$$\begin{aligned} 0 \to & \mathcal{O}_X(D_4 - 3D_6) \to \mathcal{O}_X(D_4 - 2D_6)^{\oplus 2} \oplus \mathcal{O}_X(-2D_6) \\ \to & \mathcal{O}_X(D_4 - D_6) \oplus \mathcal{O}_X(-D_6)^{\oplus 2} \to \mathcal{O}_X \to 0. \end{aligned} \tag{7}$$

Since $D_2 \cap D_4 \cap D_6 = \emptyset$, we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_2 - D_4 - D_6) \\ \to &\mathcal{O}_X(-D_2 - D_4) \oplus \mathcal{O}_X(-D_2 - D_6) \oplus \mathcal{O}_X(-D_4 - D_6) \\ \to &\mathcal{O}_X(-D_2) \oplus \mathcal{O}_X(-D_4) \oplus \mathcal{O}_X(-D_6) \to \mathcal{O}_X \to 0.\end{aligned}$$

Using (6), we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_4 - 2D_6) \to \mathcal{O}_X(-D_4 - D_6)^{\oplus 2} \oplus \mathcal{O}_X(-2D_6) \\ \to &\mathcal{O}_X(-D_4) \oplus \mathcal{O}_X(-D_6)^{\oplus 2} \to \mathcal{O}_X \to 0.\end{aligned} \tag{8}$$

Since $D_3 \cap D_4 = \emptyset$, we have an exact sequence

$$0 \to \mathcal{O}_X(-D_3 - D_4) \to \mathcal{O}_X(-D_3) \oplus \mathcal{O}_X(-D_4) \to \mathcal{O}_X \to 0.$$

Using (6), we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-2D_4 - D_5 + 2D_6) \\ \to &\mathcal{O}_X(-D_4 - D_5 + 2D_6) \oplus \mathcal{O}_X(-D_4) \to \mathcal{O}_X \to 0.\end{aligned} \tag{9}$$

Since $D_1 \cap D_5 = \emptyset$, we have an exact sequence

$$0 \to \mathcal{O}_X(-D_1 - D_5) \to \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_5) \to \mathcal{O}_X \to 0.$$

Using (6), we have an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(D_4 - D_5 - D_6) \\ \to &\mathcal{O}_X(D_4 - D_6) \oplus \mathcal{O}_X(-D_5) \to \mathcal{O}_X \to 0.\end{aligned} \tag{10}$$

(i) Tensoring $\mathcal{O}_X(-D_4 - D_5 + D_6)$ with (7), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_5 - 2D_6) \to \mathcal{O}_X(-D_5 - D_6)^{\oplus 2} \oplus \mathcal{O}_X(-D_4 - D_5 - D_6) \\ \to &\mathcal{O}_X(-D_5) \oplus \mathcal{O}_X(-D_4 - D_5)^{\oplus 2} \to \mathcal{O}_X(-D_4 - D_5 + D_6) \to 0.\end{aligned}$$

We have already known that all line bundles in the sequence except $\mathcal{O}_X(-D_4 - D_5 + D_6)$ belong to $\langle \mathfrak{D}_{\text{nef}} \rangle$. Thus so does $\mathcal{O}_X(-D_4 - D_5 + D_6)$, which means that

$$\langle \mathfrak{D}_{\text{nef}} \rangle = \langle F_{m*}\mathcal{O}_X \rangle .$$

This fact has been already observed in [BT10, Proposition 3.2].

(ii) Tensoring $\mathcal{O}_X(-D_5 + D_6)$ with (8), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_4 - D_5 - D_6) \to \mathcal{O}_X(-D_4 - D_5)^{\oplus 2} \oplus \mathcal{O}_X(-D_5 - D_6) \\ \to &\mathcal{O}_X(-D_4 - D_5 + D_6) \oplus \mathcal{O}_X(-D_5)^{\oplus 2} \to \mathcal{O}_X(-D_5 + D_6) \to 0.\end{aligned}$$

We have already known from (i) that all line bundles in the sequence except $\mathcal{O}_X(-D_5 + D_6)$ belong to $\langle F_{m*}\mathcal{O}_X \rangle$. Thus so does $\mathcal{O}_X(-D_5 + D_6)$.

(iii) Tensoring $\mathcal{O}_X(-D_4 - D_5 + 2D_6)$ with (7), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_5 - D_6) \to \mathcal{O}_X(-D_5)^{\oplus 2} \oplus \mathcal{O}_X(-D_4 - D_5) \\ \to &\mathcal{O}_X(-D_4 - D_5 + D_6)^{\oplus 2} \oplus \mathcal{O}_X(-D_5 + D_6) \to \mathcal{O}_X(-D_4 - D_5 + 2D_6) \to 0.\end{aligned}$$

We have already known from (i) and (ii) that all line bundles in the sequence except $\mathcal{O}_X(-D_4 - D_5 + 2D_6)$ belong to $\langle F_{m*}\mathcal{O}_X\rangle$. Thus so does $\mathcal{O}_X(-D_4 - D_5 + 2D_6)$.

(iv) Take $j = 1, 2$. Tensoring $\mathcal{O}_X(D_4 - jD_6)$ with (9), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(-D_4 - D_5 + (2-j)D_6) \\ \to &\mathcal{O}_X(-D_5 + (2-j)D_6) \oplus \mathcal{O}_X(-jD_6) \to \mathcal{O}_X(D_4 - jD_6) \to 0.\end{aligned}$$

We have already known from (i) and (ii) that all line bundles in the sequence except $\mathcal{O}_X(D_4 - jD_6)$ belong to $\langle F_{m*}\mathcal{O}_X\rangle$. Thus so does $\mathcal{O}_X(D_4 - jD_6)$.

(v) Take $j = 1, 2$. Tensoring $\mathcal{O}_X((1-j)D_6)$ with (10), we obtain an exact sequence

$$\begin{aligned}0 \to &\mathcal{O}_X(D_4 - D_5 - jD_6) \\ \to &\mathcal{O}_X(D_4 - jD_6) \oplus \mathcal{O}_X(-D_5 + (1-j)D_6) \to \mathcal{O}_X((1-j)D_6) \to 0.\end{aligned}$$

We have already known from (iv) that all line bundles in the sequence except $\mathcal{O}_X(D_4 - D_5 - jD_6)$ belong to $\langle F_{m*}\mathcal{O}_X\rangle$. Thus so does $\mathcal{O}_X(D_4 - D_5 - jD_6)$.

Therefore we know that $\langle \mathfrak{D}_{\mathrm{nef}} \rangle = \langle \mathfrak{D}(\omega_X^{-3})_m \rangle$. On the other hand, we can directly see by computation that

$$\mathfrak{D} \subset \mathfrak{D}(\omega_X^{-1})_m = \mathfrak{D}(\omega_X^{-2})_m = \mathfrak{D}(\omega_X^{-3})_m$$

for $i = 1, 2$, which is more or less expected by Lemma 3.5. It is also known that $\bigoplus_{i=0}^{3} \omega_X^{-i}$ is a generator ([VdB04, Lemma 3.2.2]), since ω_X^{-3} is ample. Thus we can see that

$$\bigoplus_{\mathcal{L} \in \mathfrak{D}_{\mathrm{nef}}} \mathcal{L}$$

is a generator of $D^b(X)$. Combining the result in §4.3 with Lemma 2.4, we can conclude that the set $\mathfrak{D}_{\mathrm{nef}}$ forms a full strong exceptional collection.

5.3 Exceptional collection on Fano 3-fold in (18)

Take the Fano 3-fold in (18). We use the same notation as in §4.4. We want to find all elements of $\mathfrak{D}(\omega_X^{-3})_m$ for sufficiently large m. For $\boldsymbol{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in$

P^3_m and $\boldsymbol{w} = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix} \in \mathbb{Z}^8$, we have

$$\boldsymbol{q}^m(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{pmatrix} \lfloor \frac{(x-3)+3}{m} \rfloor \\ \lfloor \frac{(y-3)+3}{m} \rfloor \\ \lfloor \frac{(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(y-3)+(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(y-3)+3}{m} \rfloor \\ \lfloor \frac{-(z-3)+3}{m} \rfloor \\ \lfloor \frac{(y-3)-(z-3)+3}{m} \rfloor \\ \lfloor \frac{-(x-3)+(y-3)+3}{m} \rfloor \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \lfloor \frac{-y+z+3}{m} \rfloor \\ \lfloor \frac{-y+6}{m} \rfloor \\ \lfloor \frac{-z+6}{m} \rfloor \\ \lfloor \frac{y-z+3}{m} \rfloor \\ \lfloor \frac{-x+y+3}{m} \rfloor \end{pmatrix}.$$

Thus we have

$$q^m_4(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } z \geq y + m - 3 \\ 0 & \text{if } y + m - 3 > z \geq y - 3 \\ -1 & \text{if } y > z + 3 \end{cases}$$

$$q^m_5(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 0 & \text{if } 6 \geq y \\ -1 & \text{if } y > 6 \end{cases}$$

$$q^m_6(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 0 & \text{if } 6 \geq z \\ -1 & \text{if } z > 6 \end{cases}$$

$$q^m_7(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } y \geq z + m - 3 \\ 0 & \text{if } z + m - 3 > y \geq z - 3 \\ -1 & \text{if } z > y + 3 \end{cases}$$

$$q^m_8(\boldsymbol{u}, -3\boldsymbol{w}) = \begin{cases} 1 & \text{if } y \geq x + m - 3 \\ 0 & \text{if } x + m - 3 > y \geq x - 3 \\ -1 & \text{if } x > y + 3. \end{cases}$$

Hence by tedious computation, we can see $\mathfrak{D}(\omega_X^{-3})_m = \mathfrak{D} \cup \mathfrak{D}'$, where

$$\begin{aligned} \mathfrak{D}' = \big\{ & \mathcal{O}_X(-D_4 - iD_8), \mathcal{O}_X(-D_5 - iD_8), \mathcal{O}_X(-D_6 - iD_8), \\ & \mathcal{O}_X(-D_7 - iD_8), \mathcal{O}_X(-D_4 - D_5 + D_7 + jD_8), \\ & \mathcal{O}_X(D_4 - D_6 - D_7 + iD_8) \;\big|\; i = -1, 0 \text{ and } j = 0, 1 \big\}. \end{aligned}$$

Note that there are linearly equivalences;

$$D_1 \sim D_8, \quad D_2 + D_4 + D_5 \sim D_7 + D_8, \quad D_3 + D_4 \sim D_6 + D_7. \tag{11}$$

We have

$$0 \to \mathcal{O}_X(-D_1 - D_8) \to \mathcal{O}_X(-D_1) \oplus \mathcal{O}_X(-D_8) \to \mathcal{O}_X \to 0.$$

Combining this with (11), we have

$$0 \to \mathcal{O}_X(-2D_8) \to \mathcal{O}_X(-D_8) \oplus \mathcal{O}_X(-D_8) \to \mathcal{O}_X \to 0.$$

By tensoring $\mathcal{L} \in \operatorname{Pic} X$, we obtain the following:

Claim 5.3. *If $\mathcal{L}$ and $\mathcal{L} \otimes \mathcal{O}_X(D_8) \in \langle \mathfrak{D} \rangle = \langle F_{m*}\mathcal{O}_X \rangle$, then we have $\mathcal{L} \otimes \mathcal{O}_X(iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle$ for all $i \in \mathbb{Z}$.*

We shall check that $\mathcal{L} \in \langle F_{m*}\mathcal{O}_X \rangle$ for all $\mathcal{L} \in \mathfrak{D}'$ below. We take an arbitrary integer $i \in \mathbb{Z}$.

(i) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_5 - D_6 - D_7 + iD_8) \to & \mathcal{O}_X(-D_6 - D_7 + iD_8) \\ & \to \mathcal{O}_{D_5}(-D_6 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_5 - D_6 + iD_8) \to \mathcal{O}_X(-D_6 + iD_8) \to \mathcal{O}_{D_5}(-D_6 + iD_8) \to 0.$$

Hence $\mathcal{O}_X(-D_6 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle$, since

$$\mathcal{O}_X(-D_5-D_6-D_7+iD_8), \mathcal{O}_X(-D_5-D_6+iD_8), \mathcal{O}_X(-D_6-D_7+iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle .$$

Similarly we obtain $\mathcal{O}_X(-D_5 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle$.

(ii) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_3 - D_4 - D_5 + iD_8) \to & \mathcal{O}_X(-D_3 - D_4 + iD_8) \\ & \to \mathcal{O}_{D_5}(-D_4 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_4 - D_5 + iD_8) \to \mathcal{O}_X(-D_4 + iD_8) \to \mathcal{O}_{D_5}(-D_4 + iD_8) \to 0.$$

Since the line bundles

$$\mathcal{O}_X(-D_3 - D_4 - D_5 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 - D_7 + iD_8),$$

$$\mathcal{O}_X(-D_3 - D_4 + iD_8) \cong \mathcal{O}_X(-D_6 - D_7 + iD_8)$$

and

$$\mathcal{O}_X(-D_4 - D_5 + iD_8)$$

belong to $\langle F_{m*}\mathcal{O}_X \rangle$ by (11), we have $\mathcal{O}_X(-D_4 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle$.

(iii) We have exact sequences:

$$\begin{aligned} 0 \to \mathcal{O}_X(-D_2 - D_6 - D_7 + iD_8) \to & \mathcal{O}_X(-D_2 - D_7 + iD_8) \\ & \to \mathcal{O}_{D_6}(-D_7 + iD_8) \to 0, \end{aligned}$$

$$0 \to \mathcal{O}_X(-D_6 - D_7 + iD_8) \to \mathcal{O}_X(-D_7 + iD_8) \to \mathcal{O}_{D_6}(-D_7 + iD_8) \to 0.$$

Since we can see from (11) that

$$\mathcal{O}_X(-D_2 - D_6 - D_7 + iD_8) \cong \mathcal{O}_X(-D_4 - D_5 - D_6 + (i+1)D_8) \in \langle F_{m*}\mathcal{O}_X \rangle ,$$

$$\mathcal{O}_X(-D_2 - D_7 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 + (i+1)D_8) \in \langle F_{m*}\mathcal{O}_X \rangle\,,$$
$$\mathcal{O}_X(-D_6 - D_7 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle\,,$$

we know that $\mathcal{O}_X(-D_7 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle$.

(iv) We have exact sequences:

$$0 \to \mathcal{O}_X(-D_2 - D_3 - D_7 + iD_8) \to \mathcal{O}_X(-D_2 - D_3 + iD_8) \to \mathcal{O}_{D_7}(-D_2 + iD_8) \to 0,$$

$$0 \to \mathcal{O}_X(-D_2 - D_7 + iD_8) \to \mathcal{O}_X(-D_2 + iD_8) \to \mathcal{O}_{D_7}(-D_2 + iD_8) \to 0.$$

Since we have

$$\mathcal{O}_X(-D_2 - D_3 - D_7 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 - D_7 + (i+1)D_8) \in \langle F_{m*}\mathcal{O}_X \rangle\,,$$
$$\mathcal{O}_X(-D_2 - D_3 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 + (i+1)D_8) \in \langle F_{m*}\mathcal{O}_X \rangle\,,$$
$$\mathcal{O}_X(-D_2 - D_7 + iD_8) \cong \mathcal{O}_X(-D_4 - D_5 + (i+1)D_8) \in \langle F_{m*}\mathcal{O}_X \rangle$$

by (11), we have

$$\mathcal{O}_X(-D_4 - D_5 + D_7 + (i+1)D_8) \cong \mathcal{O}_X(-D_2 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle\,.$$

(v) We have exact sequences:

$$0 \to \mathcal{O}_X(-D_2 - D_3 - D_4 + iD_8) \to \mathcal{O}_X(-D_3 - D_4 + iD_8) \to \mathcal{O}_{D_2}(-D_3 + iD_8) \to 0,$$

$$0 \to \mathcal{O}_X(-D_2 - D_3 + iD_8) \to \mathcal{O}_X(-D_3 + iD_8) \to \mathcal{O}_{D_2}(-D_3 + iD_8) \to 0$$

Since we have

$$\mathcal{O}_X(-D_2 - D_3 - D_4 + iD_8) \cong \mathcal{O}_X(-D_4 - D_5 - D_6 + (i+1)D_8) \in \langle F_{m*}\mathcal{O}_X \rangle\,,$$
$$\mathcal{O}_X(-D_3 - D_4 + iD_8) \cong \mathcal{O}_X(-D_6 - D_7 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle\,,$$
$$\mathcal{O}_X(D_2 - D_3 + iD_8) \cong \mathcal{O}_X(-D_5 - D_6 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle$$

by (11), we have $\mathcal{O}_X(D_4 - D_6 - D_7 + iD_8) \cong \mathcal{O}_X(-D_3 + iD_8) \in \langle F_{m*}\mathcal{O}_X \rangle$.

Hence we know that $\langle \mathfrak{D} \rangle = \left\langle \mathfrak{D}(\omega_X^{-3})_m \right\rangle$. Then by a similar argument to one given in §5.2, we can conclude that the set $\mathfrak{D}_X$ forms a full strong exceptional collection.

6 Birational contractions and tilting objects

Lemma 6.1. *Let $(f, \varphi)\colon (X, \Delta_X) \to (Y, \Delta_Y)$ be a T-equivariant extremal birational contraction between smooth projective toric varieties. Choose a maximal cone σ in Δ_X such that $\varphi(\sigma)$ is a cone in Δ_Y. For any $\boldsymbol{u} \in P_m^n$,*

we denote a divisor $D^X_{\boldsymbol{u},\mathbf{0},\sigma}$ *on* X *(resp.* $D^Y_{\boldsymbol{u},\mathbf{0},\varphi(\sigma)}$ *on* Y*) by* $D^X_{\boldsymbol{u}}$ *(resp.* $D^Y_{\boldsymbol{u}}$*). Then we have* $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}}) = \mathcal{O}_Y(D^Y_{\boldsymbol{u}})$*. In particular,*

$$\mathfrak{D}_Y = \{ f_*\mathcal{L}_X \mid \mathcal{L}_X \in \mathfrak{D}_X \}.$$

and

$$\mathcal{O}_X(D^X_{\boldsymbol{u}}) = f^*\mathcal{O}_Y(D^Y_{\boldsymbol{u}}) \otimes \mathcal{O}_X(aE),$$

where $a \geq 0$ *and* E *is the exceptional divisor of* f*.*

Proof. From the commutativity $F_m \circ f = f \circ F_m$, we obtain

$$\bigoplus_{\boldsymbol{u}\in P^n_m} \mathcal{O}_Y(D^Y_{\boldsymbol{u}}) = F_{m*}\mathcal{O}_Y = F_{m*}f_*\mathcal{O}_X = f_*F_{m*}\mathcal{O}_X = \bigoplus_{\boldsymbol{u}\in P^n_m} f_*\mathcal{O}_X(D^X_{\boldsymbol{u}}).$$

Since all $\mathcal{O}_Y(D^Y_{\boldsymbol{u}})$ and $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}})$ are indecomposable and $\operatorname{End}_Y(\mathcal{O}_Y(D^Y_{\boldsymbol{u}})) = \mathbb{C}$, for $\boldsymbol{u} \in P^n_m$ we have $\boldsymbol{u}' \in P^n_m$ such that $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}}) = \mathcal{O}_Y(D^Y_{\boldsymbol{u}'})$. On the other hand, we have an inclusion $f_*\mathcal{O}_X(D^X_{\boldsymbol{u}}) \hookrightarrow \mathcal{O}_Y(D^Y_{\boldsymbol{u}})$, which is isomorphic in codimension one. Thus it is isomorphic. □

Lemma 6.2. *In the situation of Lemma 6.1, assume that* f *is a equivariant blowing-up along a* T*-invariant smooth center* C*, and define* $d := \dim E - \dim C$*,* $n := \dim X$ *and* $\mathcal{L}_Y := f_*\mathcal{L}_X$ *for some* $\mathcal{L}_X \in \mathfrak{D}_X$*. Consider the Leray spectral sequence*

$$\begin{aligned} E_2^{p,q} =& H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_*\mathcal{O}_X((ma+d)E)) \\ \Longrightarrow E^{p+q} =& H^{p+q}(X, f^*(\mathcal{L}_Y^{\otimes m} \otimes \omega_X) \otimes \mathcal{O}_X((ma+d)E)) \end{aligned}$$

and assume furthermore that the vanishing

$$\operatorname{Hom}^i_X(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = 0 \tag{12}$$

holds for all $i > 0$*.*

(i) The vanishing

$$\operatorname{Hom}^i_Y(\mathcal{L}_Y, F_{m*}\mathcal{O}_Y) = 0$$

holds for all $i > 0$ *if and only if* $E_2^{p,d} = 0$ *for all* $p < n-d-1 = \dim C$*.*

In particular, if $d = n-1$*, namely if* C *is a point, this is automatically true.*

(ii) Assume that

$$H^i(C, \mathcal{L}_Y^{\otimes -m} \otimes f_*\mathcal{O}_E(l-d-1)) = 0 \tag{13}$$

for $i > 0$ *and all* l *with* $ma + d \geq l$*, where we define* $\mathcal{O}_E(1)$ *to be the tautological line bundle of the* $\mathbb{P}^d$*-bundle* $E \to C$*. Then* $E_2^{p,d} = 0$ *for all* p *with* $p < n - d - 1$*.*

Proof. (i) First of all, we have

$$\begin{aligned}
&\mathrm{Hom}^i_X(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = \mathrm{Hom}^i_X(F_m^*\mathcal{L}_X, \mathcal{O}_X)\\
=&H^{n-i}(X, \mathcal{L}_X^{\otimes m} \otimes \omega_X)^\vee = H^{n-i}(X, \mathcal{L}_X^{\otimes m} \otimes f^*\omega_Y \otimes \mathcal{O}_X(dE))^\vee\\
=&H^{n-i}(X, f^*(\mathcal{L}_Y^{\otimes m} \otimes \omega_X) \otimes \mathcal{O}_X((ma+d)E))^\vee\\
=&(E^{n-i})^\vee.
\end{aligned}$$

Hence (12) means that

$$E^{p+q} = 0 \tag{14}$$

for all $p + q \neq n$. Similarly it is easy to see that

$$(E_2^{n-i,0})^\vee = \mathrm{Hom}^i_Y(\mathcal{L}_Y, F_{m*}\mathcal{O}_Y).$$

Therefore what we have to show is that, under assuming (14), $E_2^{p,0} = 0$ for all $p < n$ is equivalent to $E_2^{p,d} = 0$ for all $p < n - d - 1$. More strongly, we will see below $E_2^{p,d} \cong E_2^{p+d+1,0}$ for $p < n - d - 1$.

Note that

$$\mathbb{R}^q f_*\mathcal{O}_E(lE) = 0 \tag{15}$$

unless $q = 0, d$, since $f|_E \colon E \to C$ is a $\mathbb{P}^d$-bundle. We also have

$$f_*\mathcal{O}_E(lE) = 0$$

for all positive l. Then we have a short exact sequence

$$\begin{aligned}
0 \to \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_*\mathcal{O}_X((l-1)E) \to&\mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_*\mathcal{O}_X(lE)\\
\to&\mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_*\mathcal{O}_E(lE) \to 0
\end{aligned} \tag{16}$$

for $l \geq 0$ and all q. Hence by the vanishing $\mathbb{R}^q f_*\mathcal{O}_X = 0$ for $q \neq 0$ and (15), we conclude that

$$\begin{aligned}
E_2^{p,q} =&H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_*\mathcal{O}_X((ma+d)E))\\
\cong&H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_*\mathcal{O}_X((ma+d-1)E))\\
\cong&\cdots\\
\cong&H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^q f_*\mathcal{O}_X) = 0
\end{aligned}$$

for all p and all $q \neq 0, d$. Thus we have $E_2^{p,q} \cong E_{d+1}^{p,q}$ for all p, q. Therefore from (14) we obtain

$$E_2^{p,d} \cong E_{d+1}^{p,d} \cong E_{d+1}^{p+d+1,0} \cong E_2^{p+d+1,0}$$

for $p + d + 1 < n$. Thus we obtain the conclusion.

(ii) By the duality,

$$\begin{aligned}
&H^{n-d-1-p}(C, \mathcal{L}_Y^{\otimes -m} \otimes f_* \mathcal{O}_E(l-d-1))^\vee \\
=&H^p(C, \mathcal{L}_Y^{\otimes m} \otimes (f_* \mathcal{O}_E(l-d-1))^\vee \otimes \omega_C) \\
=&H^p(Y, \mathcal{L}_Y^{\otimes m} \otimes \omega_Y \otimes \mathbb{R}^d f_* \mathcal{O}_E(lE)).
\end{aligned}$$

By the assumption (13), the last one vanishes for all l, p with $ma + d \geq l$ and $p < n - d - 1$. Then the vanishing of $E_2^{p,d}$ is a direct consequence of the vanishing $\mathbb{R}^d f_* \mathcal{O}_X = 0$ and (16). □

Theorem 6.3. *Let X be a toric del Pezzo surface. Then $\mathfrak{D}_X$ is a full strong exceptional collection on X.*

Proof. We have already checked the statement for the maximal del Pezzo surface Y_3 in §4.2. Then the statement for the other cases follows from Lemmas 5.1 and 6.2. □

See also [HP08, Theorem 8.2] for an interesting result in this direction.

Lemma 6.4. *In the notation in Lemma 6.2, assume that X and Y are Fano 3-folds and that the vanishing*

$$\mathrm{Hom}_X^i(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = 0 \tag{17}$$

holds for all $i > 0$. Then the vanishing

$$\mathrm{Hom}_Y^i(\mathcal{L}_Y, F_{m*}\mathcal{O}_Y) = 0$$

holds for all $i > 0$.

Proof. We divide the proof into two parts.

Step 1. By the last assertion in Lemma 6.2(i), we may assume that $C \cong \mathbb{P}^1$. There are primitive generators $\boldsymbol{v}_1, \dots, \boldsymbol{v}_5$ of 1-dimensional cones in Δ_Y (and we sometimes regard them as generators of 1-dimensional cones in Δ_X) such that

- C is the T-invariant curve corresponding to the 2-dimensional cone generated by $\boldsymbol{v}_1$ and $\boldsymbol{v}_4$, and
- the sets $\{\boldsymbol{v}_1, \boldsymbol{v}_4, \boldsymbol{v}_5\}$, $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_4\}$ and $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ generate 3-dimensional cones in Δ_Y respectively.
- $\boldsymbol{v}_3$ is different from $\boldsymbol{v}_4$, but it may coincide with $\boldsymbol{v}_5$.

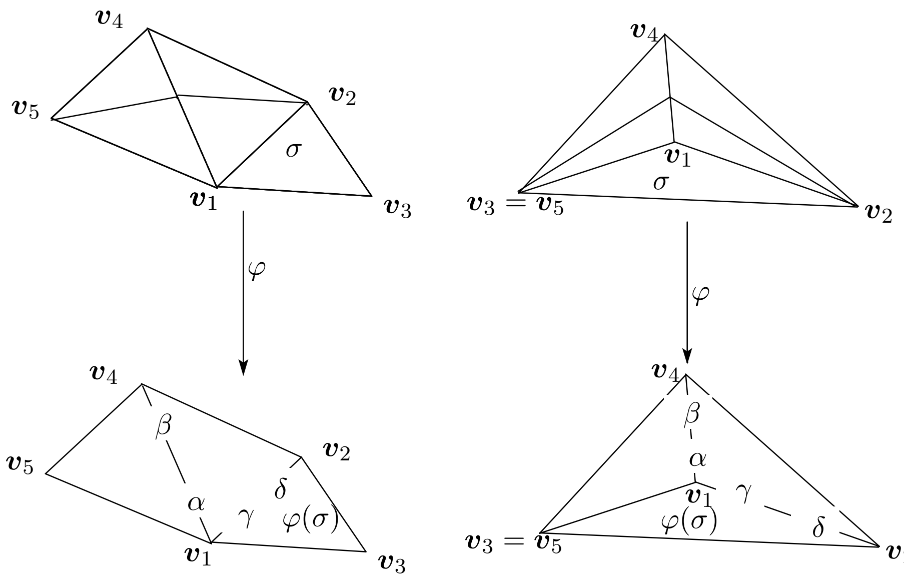

Figure 6: Divisorial contractions

We have the following equalities

$$\boldsymbol{v}_2 + \boldsymbol{v}_5 + \alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_4 = \mathbf{0} \quad \text{and} \quad \boldsymbol{v}_3 + \boldsymbol{v}_4 + \gamma \boldsymbol{v}_1 + \delta \boldsymbol{v}_2 = \mathbf{0}.$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ (see Figure 6). Without the loss of generality, we may assume that $\beta \geq \alpha$. Then we know that $\beta \geq 0$, since $\alpha + \beta \geq -1$ by the condition that Y is Fano [Od88, Page 89]. By these equalities, we have

$$\boldsymbol{v}_5 - \beta \boldsymbol{v}_3 + (1 - \beta\delta)\boldsymbol{v}_2 + (\alpha - \beta\gamma)\boldsymbol{v}_1 = 0. \tag{18}$$

Note that the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$ also generates a 3-dimensional cone, say σ, in Δ_X. So we can apply Lemma 6.1 for σ. Take $\boldsymbol{u} \in P^3_m$ such that $\mathcal{L}_Y \cong \mathcal{O}_X(D^Y_{\boldsymbol{u},\varphi(\sigma)})$, and denote by $D_i (= D^Y_i)$ (resp. D^X_i) the prime divisors on Y (resp. X) corresponding to $\boldsymbol{v}_i$, and we put q_i to be the coefficient of D_i in a T-invariant divisor $D^Y_{\boldsymbol{u},\varphi(\sigma)}$ on Y, namely we have

$$D^Y_{\boldsymbol{u},\varphi(\sigma)} = q_1 D_1 + q_2 D_2 + q_3 D_3 + q_4 D_4 + \cdots .$$

We have

$$D^X_{\boldsymbol{u},\sigma} = f^* D^Y_{\boldsymbol{u},\varphi(\sigma)} + aE, \quad \text{that is} \quad \mathcal{L}_X \cong f^*\mathcal{L}_Y \otimes \mathcal{O}_X(aE)$$

for some $a \geq 0$ as in Lemma 6.1.

To check (13) in the case $C \cong \mathbb{P}^1$, it is enough to show

$$H^1(C, \mathcal{L}_Y^{\otimes -m} \otimes f_*\mathcal{O}_E(ma-1)) = 0. \tag{19}$$

By choosing $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ as a basis of the lattice $N \cong \mathbb{Z}^3$, we obtain from (18) that

$$\boldsymbol{v}_5 = {}^t(\beta\gamma - \alpha, \beta\delta - 1, \beta) \tag{20}$$

and $q_1 = q_1^m(\boldsymbol{u}, \varphi(\sigma)) = 0,$ and $q_2 = q_2^m(\boldsymbol{u}, \varphi(\sigma)) = 0$. We also know by (2) that $\mathcal{N}_{C/Y} \cong \mathcal{O}_C(\alpha) \oplus \mathcal{O}_C(\beta)$. In particular, we have

$$f_* \mathcal{O}_E(i) \cong Sym^i \mathcal{N}_{C/Y}^{\vee} \cong \mathcal{O}_C(-i\alpha) \oplus \mathcal{O}_C(-(i-1)\alpha - \beta) \oplus \cdots \oplus \mathcal{O}_C(-i\beta)$$

for $i \geq 0$. We denote F a fiber of $\mathbb{P}^1$-bundle $f|_E \colon E \to C$. Then we note that T-invariant prime divisors on X which intersect with F are only D_1^X, D_2^X, D_4^X and D_5^X (and of course, F is contained in E). Thus we have

$$-a = aE \cdot F = (D^X_{\boldsymbol{u},\sigma} - f^* D^Y_{\boldsymbol{u},\varphi(\sigma)}) \cdot F = D^X_{\boldsymbol{u},\sigma} \cdot F = q_4,$$

since $D_2^X \cdot F = D_5^X \cdot F = 0$ and $q_1 = 0$. Combining this with

$$\deg \mathcal{L}_Y|_C = (q_4 D_4 + q_5 D_5) \cdot C = \beta q_4 + q_5,$$

we have

$$\begin{aligned} &\deg \mathcal{L}_Y^{\otimes -m} \otimes O_C((1-ma)\beta) = -m(\beta q_4 + q_5) + (1 - ma)\beta \\ =& - m(\beta q_4 + q_5) + (1 + m q_4)\beta = -m q_5 + \beta. \end{aligned}$$

Since $\beta \geq \alpha$ and $\beta \geq 0$, we know that $q_5 \leqq 0$ if and only if (19) is true for $m \gg 0$.

By (20), we have

$$q_5 = \lfloor \frac{(\beta\gamma - \alpha)x + (\beta\delta - 1)y + \beta z}{m} \rfloor$$

for $\boldsymbol{u} = {}^t(x, y, z) \in P_m^3$. By observing Figure 4, we can see that

- in all cases, we have $\beta\delta - 1 \leqq 0$,
- if $\beta \geq 2$, X is in (11) and Y is in (4) in Theorem 3.1,
- if $\beta = 1$, $\beta\gamma - \alpha \leqq 0$, and
- if $\beta \leqq 0$ (then actually $\beta = 0$), $\beta\gamma - \alpha \leqq 1$.

Consequently, we obtain $q_5 \leqq 0$, except the case X is in (11) and Y is in (4).

Step 2. Let X be the Fano 3-fold in (11) and take $\mathcal{L}_X \in \mathfrak{D}_X$. Then we have seen in §4.3 that $\mathcal{L}_X \not\cong \mathcal{O}_X(-D_4 - D_5 + D_6)$ if and only if the equality (17) holds for all $i > 0$. Note that $\boldsymbol{v}_6$ in Figure 5 plays the role of $\boldsymbol{v}_5$ in Figure 6. Consequently we know that if $\mathcal{L}_X$ is not isomorphic to $\mathcal{O}_X(-D_4 - D_5 + D_6)$, then $q_5 \leq 0$ by the computation in §4.3. Then the result follows. $\square$

Now we give the proof of Theorem 1.1.

Proof. In §5 we have already seen that $F_{m*}\mathcal{O}_X$ is a generator for maximal smooth toric Fano 3-folds X. We have also seen in §5.1 and 5.3 that $F_{m*}\mathcal{O}_X$ is tilting for the Fano 3-folds in (17) and (18). Then Lemmas 5.1 and 6.4 imply that $\mathfrak{D}_X$ is a full strong exceptional collection for all smooth toric Fano 3-folds except the cases (4) and (11).

For the case X in (11), we have seen in §5.2 that the set $\mathfrak{D}_{\text{nef}}$ is a full strong exceptional collection on X, and in §4.3 that

$$\mathrm{Hom}^i_X(\mathcal{L}_X, F_{m*}\mathcal{O}_X) = 0$$

holds for all $i > 0$ and all $\mathcal{L}_X \in \mathfrak{D}_{\text{nef}}$. Take the Fano 3-fold Y in (4) and consider the blowing-up $f\colon X \to Y$ in Fugure 3. Then Lemmas 5.1 and 6.4 implies that the subset $\left\{ f_*\mathcal{L}_X \;\middle|\; \mathcal{L}_X \in \mathfrak{D}_{\text{nef}} \right\}$ of $\mathfrak{D}_Y$ forms a full strong exceptional collection on Y.

$\square$

7 Derived equivalences of flops

The following example should be generalized a lot, but in this article we do not pursue its generalization.

Let X be the Fano 3-fold in (18). Then the prime divisor D_2 in the notation in §4.4 is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and there are extremal birational contractions $f^+\colon X \to X^+, f^-\colon X \to X^-$ which contract D_2 in two different directions respectively. X^+ and X^- are connected by a standard flop, and several people, including Bondal, Orlov, Bridgeland and Van den Bergh, observe that they are derived equivalent. We shall prove it by using tilting generators we constructed above.

We can see $F_{m*}\mathcal{O}_{X^+}$ and $F_{m*}\mathcal{O}_{X^-}$ are tilting generators on $D^b(X^+)$ and $D^b(X^-)$ respectively by a similar idea to Lemma 6.4. Moreover the former is the strict transform of the later. Define the endmorphism algebras as

$$A^+ = \mathrm{End}_{X^+}(F_{m*}\mathcal{O}_{X^+}), \quad A^- = \mathrm{End}_{X^-}(F_{m*}\mathcal{O}_{X^-}).$$

Then because X^+ and X^- are isomorphic in codimension 1, we have a natural isomorphism $A^+ \cong A^-$. Hence

$$D^b(X^+) \cong D^b(\mathrm{mod}\, A^+) \cong D^b(\mathrm{mod}\, A^-) \cong D^b(X^-).$$

References

[Ba82] V. Batyrev, Toroidal Fano 3-folds, Math. USSR-Izv. 19, 13-25, 1982.

[Ba04] A. Bayer, Semisimple quantum cohomology and blowups, Int. Math. Res. Not. 2004, no. 40, 2069–2083.

[BT10] A. Bernardi, S. Tirabassi, Derived Categories of Toric Fano 3-Folds via the Frobenius Morphism, arXiv:1002.1666.

[Bo06] A. I. Bondal, Derived categories of toric varieties. Obervolfach Reports, 3 (1), 284–286, 2006.

[BV03] A. I. Bondal, M. Van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, Mosc. Math. J. 3 (2003), 1-36.

[BH09] L. Borisov, Z. Hua, On the conjecture of King for smooth toric Deligne-Mumford stacks, Adv. Math 221, (2009), 277-301.

[CM10] L. Costa, R.M. Miro-Roig, Frobenius splitting and derived category of toric varieties, arXiv:1006.5315, to appear in Illinois Journal of Mathematics .

[CM] L. Costa, R.M. Miro-Roig, Derived category of toric varieties with small Picard numbers, preprint.

[CRM09] L. Costa, S. Di Rocco, R.M. Miro-Roig, Derived category of toric fibrations, arXiv:0908.0846.

[Cr09] A. Craw, Quiver flag varieties and multigraded linear series, to appear in Duke Math Journal.

[DLM10] A. Dey, M. Lason, M. Michalek, Derived category of toric varieties with Picard number three, arXiv:1002.3591.

[Fu93] W. Fulton, Introduction to toric varieties. Annals of Mathematics Studies, 131. Princeton University Press, Princeton, NJ, 1993. xii+157 pp.

[HP06] L. Hille, M. Perling, A Counterexample to King's Conjecture Compos. Math. 142, 1507-1521 (2006).

[HP08] L. Hille, M. Perling, Exceptional Sequences of Invertible Sheaves on Rational Surfaces, arXiv:0810.1936.

[Hu06] D. Huybrechts, Fourier-Mukai transforms in algebraic geometry. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2006. viii+307 pp.

[IU09] A. Ishii, K. Ueda, Dimer models and exceptional collections, arXiv:0911.4529.

[Ka06] Y. Kawamata, Derived Categories of Toric Varieties, Michigan Math. J. 54 (2006), no. 3, 517–535.

[Ki] A. King. Tilting bundles on some rational surfaces, preprint.

[LM10] M. Lason, M. Michalek, On the full, strongly exceptional collections on toric varieties with Picard number three, arXiv:1003.2047.

[Od88] T. Oda, Convex bodies and algebraic geometry. An introduction to the theory of toric varieties. Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 15. Springer-Verlag, Berlin, 1988. viii+212 pp.

[Sa09] A. Samokhin, Tilting bundles via the Frobenius morphism, arXiv:0904.1235v2.

[Th00] Jesper F. Thomsen, Frobenius direct images of line bundles on toric varieties. J. Algebra. 226 (2000), 865–874.

[VdB04] M. Van den Bergh, Three-dimensional flops and noncommutative rings. Duke Math. J. 122 (2004), 423–455.

[WW82] K. Watanabe, M. Watanabe, The classification of Fano 3-folds with torus embeddings, Tokyo J. Math. 5, (1982), 37-48.

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